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LARGE AMPLITUDE PATTERNS FOR TWO COMPETING SPECIES

Robert A. Gardner

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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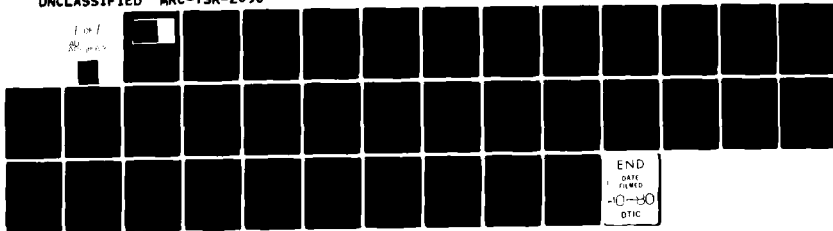LARGE AMPLITUDE PATTERNS FOR
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ABSTRACT

Large amplitude solutions are obtained for systems of semilinear reaction-diffusion equations arising in mathematical ecology which describe the evolution of two competing species. Their behavior is locally consistent with the principle of competitive exclusion. Such solutions are first obtained for a special class of steady state equations in which the two species are assumed to be exactly equal competitors; large amplitude patterns for generic classes of equations are then obtained by introducing various perturbations in the relative competitive strengths of the two species. In particular, we obtain (1), travelling wave solutions through constant perturbations, and (2), stable stationary solutions through spatially inhomogeneous perturbations.

SIGNIFICANCE AND EXPLANATION

Reaction-diffusion systems are systems of nonlinear partial differential equations which arise in various aspects of science and engineering, including mathematical ecology. Such equations can be used to describe the evolution of interacting and diffusing species. An important problem is to determine the eventual behavior of their solutions. It is therefore of interest to study the existence and stability properties of certain distinguished solutions, or patterns, such as equilibria and travelling waves. Such results are still quite fragmentary, especially with regard to large amplitude phenomena. The purpose of this paper is to study such questions for a specific system arising in mathematical ecology which describes the evolution of two competing species, in the hope that in so doing we shall develop techniques that will prove useful in studying more general classes of equations.

LARGE AMPLITUDE PATTERNS FOR
TWO COMPETING SPECIES

Robert A. Gardner

0. Introduction

The topic discussed here is the existence of large amplitude solutions of reaction-diffusion systems arising in mathematical ecology which describe the evolution of two competing species. We seek solutions which are locally consistent with the principle of competitive exclusion. In particular, we obtain under certain conditions, stationary solutions which partition space into regions in which one species' density, u, is near its carrying capacity while the density, v, of its competitor is small. At the boundary of adjacent regions, there is a sharp "transition layer" connecting these two states. In order to obtain such solutions, we must impose a nongeneric hypothesis which implies that the two species are exactly equal competitors. Such solutions, are therefore structurally unstable. However, by introducing various perturbations into the equations we are able to obtain several patterns which are most likely stable.

We first introduce a constant perturbation in the relative competitive strengths of the two species, so that one species u, is now a slightly stronger competitor than the other, v. In this case, we obtain a travelling wave connecting the rest state $(u,v) = R$ at $x = -\infty$, in which u excludes v, to the rest state $(u,v) = P$ at $x = +\infty$, in which v excludes u. In particular, (u,v) is a function of the variable $\zeta = x + \theta t$, and the wave moves from left to right with increasing t. We do not discuss the stability properties of such solutions here; this will be the subject of a forthcoming paper.

We next introduce spatially inhomogeneous perturbations. If, for example, it is assumed that u (resp. v) is the stronger competitor for $x < 0$, (resp. $x > 0$), we obtain a large amplitude stationary solution connecting the state R at

$x = -\infty$ to the state P at $x = +\infty$. In this case, we are able to show that such a solution, (or <u>cline</u>), is both structurally and exponentially asymptotically stable.

A number of different techniques are employed to obtain the results described above. First, we use the theory of isolating neighborhoods and the generalized Morse index to obtain certain bounded, nontrivial solutions of a broad class of equations. However, such topological arguments yield only crude information about qualitative behavior. In order to obtain finer information, we must restrict the discussion to a particular (Hamiltonian) class of equations. To study such systems, we combine the above techniques with the theory of bifurcation from simple eigenvalues. Finally, travelling wave solutions and clines are obtained by linearizing the appropriate equations about a large amplitude stationary solution, and by applying the implicit function theorem.

<u>Acknoweldgement.</u>

1. <u>Formulation of the problem.</u>

The equations to be considered are of the form

$$(1) \qquad \begin{cases} 0 = u_{xx} + f(u,v) \\ 0 = v_{xx} + g(u,v) \end{cases}$$

and

$$(2) \qquad \begin{cases} u_t = u_{xx} + f(u,v) \\ v_t = v_{xx} + g(u,v) \ , \end{cases}$$

for $x \in \mathbb{R}^1$. We shall assume that f and g have the form $f(u,v) = u\,M(u,v)$ and $g(u,v) = v\,N(u,v)$. The functions M and N are the local growth rates of u and v respectively. The interaction is characterized by the following

hypotheses;

(i) $M(u,v) \leq 0$ (resp. $N(u,v) \leq 0$) if u (resp. v) is sufficiently large;

(ii) $M_v(u,v) \leq 0$ and $N_u(u,v) \leq 0$, for all $u,v \geq 0$.

Condition (i) is a resource limitation condition, whereas condition (ii) characterizes the competitive nature of the interaction. Competition equations have been studied by a number of authors; see for example, Hirsch and Smale, [8], Conway and Smoller, [5], Gardner [6], McGehee and Armstrong, [11], Getz, [7], and May, [13]. The above hypotheses include a rather broad class of equations. For definiteness, we shall consider only those fields (uM,vN) with phase diagrams as in Figures l(a) and l(b). We shall refer to these as case (a) and

(a)   (b)

Figure 1

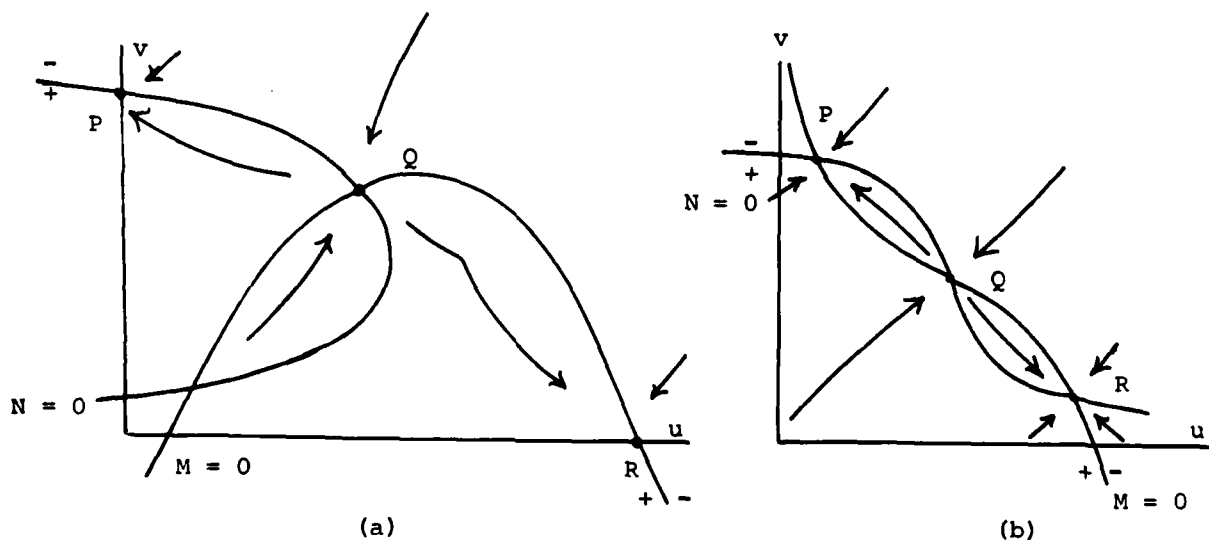case (b), respectively. Both cases satisfy (i) and (ii); case (a) has been studied by Conway and Smoller, [5], and Gardner, [6]; case (b) has been studied by Getz, [7]. Explicit examples of such systems can be found by letting

(3a) $M = -v^2 + \delta_1(u-c)(d-u)$, $N = -u^2 + \delta_2(v-e)(f-v)$

(3b) $M_\gamma = 2 + \gamma - uv^3 - u$, $N_\rho = 2 + \rho - u^3v - v$,

in cases (a) and (b), respectively. Here $\delta_1, \delta_2, d,$ and f are positive

constants, and c, e, γ, ρ are constants such that $c < d$ and $e < f$.

It will be convenient to denote (u,v) by U and $(f(U), g(U))$ by $F(U)$.

Solutions of (1) will also satisfy a first order system of the form

$$(4) \qquad \begin{cases} \dot{U} = A = (a,b) \\ \dot{A} = -F(U) \ , \end{cases}$$

or $\dot{X} = \Phi(X)$, where $X = (U,A)$, and Φ is the right hand side of (4). If

$P, Q,$ and R are the rest points of F indicated in Figure 1, denote $(P,0,0)$,

$(Q,0,0)$, and $(R,0,0)$ by $\tilde{P}, \tilde{Q},$ and $\tilde{R}$, respectively. We shall ultimately obtain

solutions of (4) which satisfy the auxilliary conditions

$$(5) \qquad \lim_{|x| \to \infty} X(x) = \tilde{R} \ ,$$

or

$$(6) \qquad \lim_{x \to -\infty} X(x) = \tilde{R} \ , \quad \lim_{x \to +\infty} X(x) = \tilde{P}.$$

Such solutions are called <u>homoclinic</u> and <u>heteroclinic</u> orbits, respectively.

We shall also obtain travelling wave solutions of (2); that is, solutions for

which U depends on the single variable $\xi = x + \theta t$. The system (2) then

reduces to a system of o.d.e.'s of the form

$$(7) \qquad U'' + F(U) - \theta U' = 0 \ ;$$

(here "prime" is $d/d\xi$). We shall seek solutions of (7) which satisfy the

condition

$$(6)_\xi \qquad \lim_{\xi \to -\infty} (U,U') = \tilde{R} \ , \quad \lim_{\xi \to +\infty} (U,U') = \tilde{P} \ .$$

2. <u>Existence of bounded, large amplitude stationary solutions</u>.

In this section, we shall obtain bounded, nontrivial solutions of (4) which

have one of the rest points $\tilde{P}$ or $\tilde{R}$ in their ω-limit sets. The technique

is to find an appropriate isolating neighborhood η for the flow of (4) so that

the index of the maximal invariant set in η is well defined. This index is

easily computed, and it can be used to obtain the existence of the solutions

described above. A familiarity with the theory of isolated invariant sets will be assumed. A concise, nontechnical description can be found in Conley and Smoller, [4]. A thorough development can be found in Conley, [3]. With the exception of the first two lemmas, the material in subsequent sections is independent of the results of this section.

LEMMA 2.1. _Suppose that_ (U,A) _is a solution of_ (4) _such that_ $|U(x)| = (u^2(x) + v^2(x))^{\frac{1}{2}} < K$ _for all_ $x \in \mathbb{R}^1$. _Then there exists_ $L > 0$ _which depends only on_ K _and_ $\sup\limits_{|U|<K} |F(U)|$ _such that_ $|A(x)| < L$ _for all_ $x \in \mathbb{R}^1$.

Proof.   This can be obtained from the Agmon-Douglis-Nirenberg estimates by viewing U as the solution of (1) on a unit interval $I = (x_0, x_0+1)$ with inhomogeneous Dirichlet conditions. In this manner, we obtain an $H^2(I)$ bound for U independently of x_0 , which in turn yields a uniform bound for A on I by the Sobolev lemma.   □

Now let $\eta^{\alpha}_{c,L} = \Sigma^{\alpha}_c \times \sigma_L$, where $\alpha = a$ or b , $\sigma_L = \{A \in \mathbb{R}^2 : |A| \leq L\}$ and Σ^{α}_c , $\alpha = a,b,$ are the regions in the U-plane indicated in Figures 2(a) and (2b), respectively.

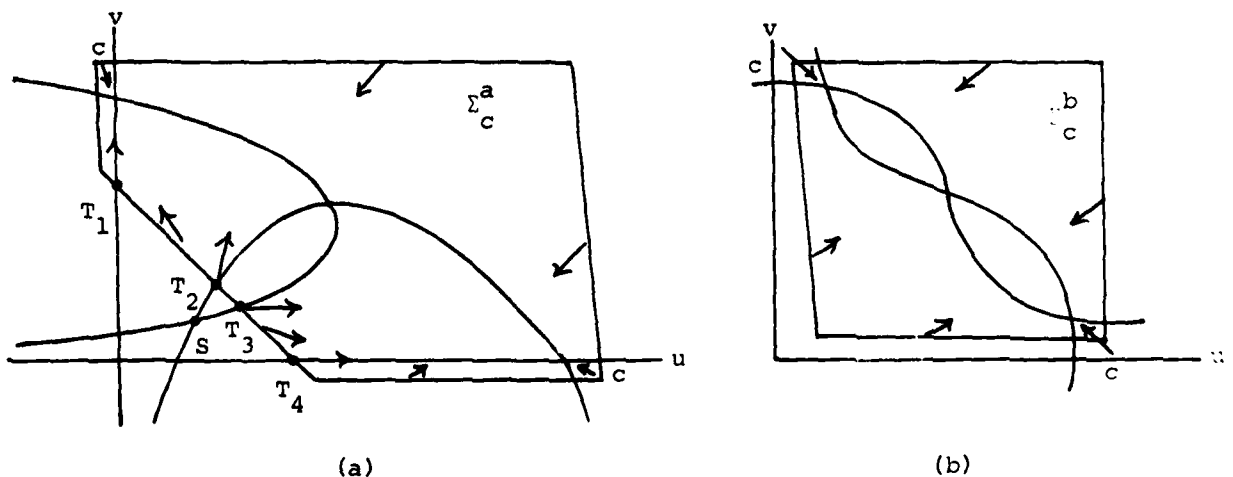(a)   (b)

Figure   2

LEMMA 2.2. _Suppose that_ L _and_ $K = \sqrt{2} c$ _are as in Lemma 2.1. Then_ $\eta_{c,L}^{b}$ _is an isolating neighborhood for (4) in case (b);_ $\eta_{c,L}^{a}$ _is an isolating neighborhood for (4) in case (a) provided that the rest point_ S _of_ F _(see Figure 2(a)) is sufficiently near the origin if it lies in the first quadrant, and that in this case, the tangent at_ S _to_ M = 0 _(resp._ N = 0 _) is close enough to a vertical (resp. horizontal) line._

Proof. We have that $\partial\eta_{c,L}^{\alpha} = \partial\Sigma_{c}^{\alpha} \times \sigma_{L} \cup \Sigma_{c}^{\alpha} \times \partial\sigma_{L}$, $\alpha = a,b$. Let (U,A) be the solution of (4) with data $(U_{0},A_{0}) \in \partial\eta_{c,L}^{\alpha}$. If $|A_{0}| = L$, then by Lemma 2.1, $U(x)$ must exit from $\Sigma_{c,L}^{\alpha}$ at some value of x. Now suppose that $U_{0} \in \partial\Sigma_{c}^{\alpha}$. It is easily verified that the vector field $-F(U)$ points strictly out of Σ_{c}^{α} along $\partial\Sigma_{c}^{\alpha}$. This is obvious in case (b). If in case (a) S lies in the first quadrant, the additional assumptions in the statement of the lemma guarantee that this will be the case. The proof is completed by expanding $U(x)$ in a Taylor series about $x = 0$. Since $U''(0) = -F(U_{0})$ points strictly out of Σ_{c}^{α}, it follows that $U(x)$ is either transverse to $\partial\Sigma_{c}^{\alpha}$ or externally tangent to $\partial\Sigma_{c}^{\alpha}$ at $x = 0$. □

The relationship between case (a) and case (b) is developed in the following lemma.

LEMMA 2.3. _The flow of (4), case (a) in_ $\eta_{c,L}^{a}$ _is related by continuation to the flow of (4), case (b) in_ $\eta_{c,L}^{b}$.

Proof. We must embed the two flows in a family of equations $\dot{X} = \Phi_{\lambda}(X)$, $0 \leq \lambda \leq 1$, where $\Phi_{0} = \Phi$ (case (a)), $\Phi_{1} = \Phi$ (case (b)), and such that there exists an isolating neighborhood η_{λ} for the λth equation which is also an isolating neighborhood for the μth equation for μ sufficiently near λ. Finally, we require that $\eta_{0} = \eta_{c,L}^{a}$ and that $\eta_{1} = \eta_{c,L}^{b}$.
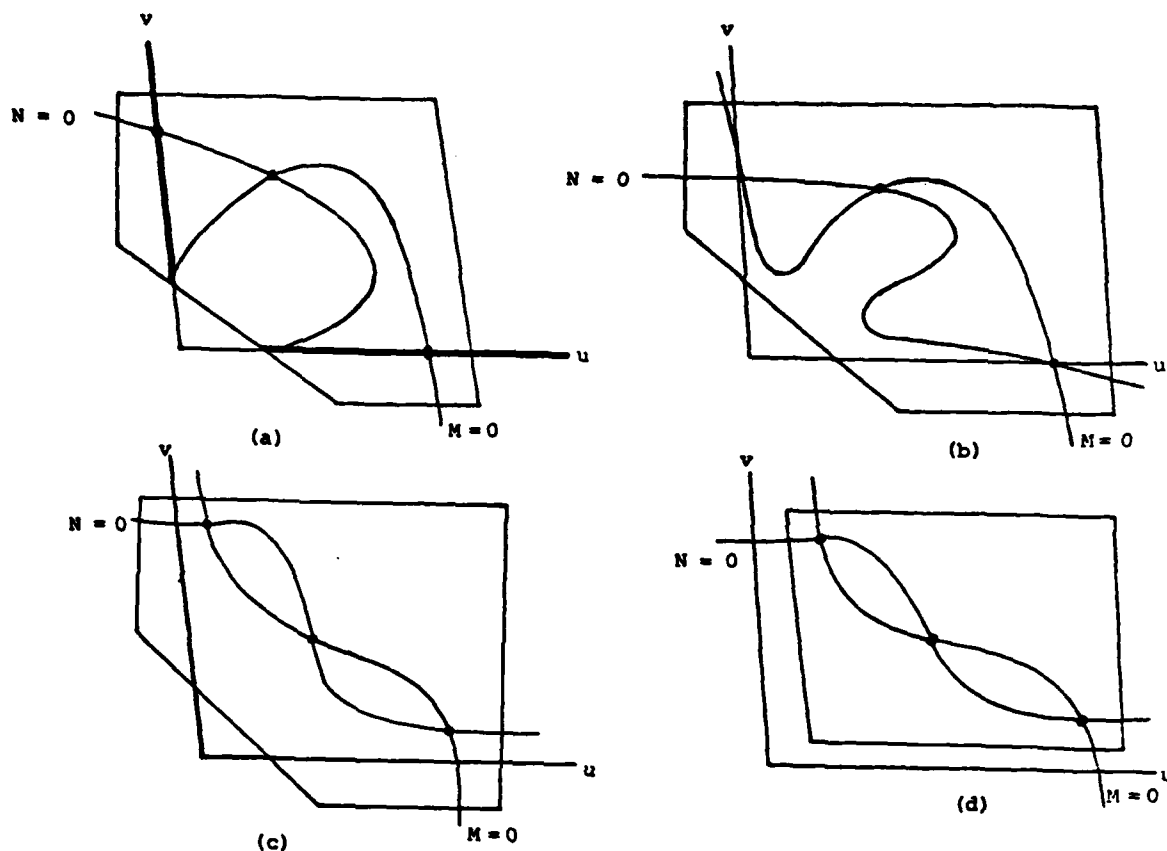
Figure 3

We begin by deforming the zero sets in Figure (2(a) to those in Figure 3 (a) by pushing T_2 (resp. T_3) along the diagonal edge of Σ_c^a until it meets T_1 (resp. T_4). Since $u = 0$ (resp. $v = 0$) is part of the zero set of f (resp. g), the phase diagram in Figure 3(a) is topologically equivalent to that of Figure 2(b), so that we can deform Figure 3(a) to Figure 3(c) as indicated. It is easily seen that these deformations can be performed in a manner such that $\eta_{c,L}^a$ is an isolating neighborhood for the flow defined by each set of equations in the above deformation.

Since the zero sets of Figure 3(c) no longer include the coordinate axes, it is easily seen that $\eta_{c,L}^{a}$ can be deformed to $\eta_{c,L}^{b}$ through a family of isolating neighborhoods $\Sigma_{\lambda} \times \sigma_{L}$. Finally, since the zero sets of Figure 3(d) coincide exactly with those of Figure 2(b) in Σ_{c}^{b}, we can deform the vector field in Figure 3(d) to the field in Figure 2(b), in a manner which leaves the zero sets fixed throughout the deformation. $\square$

If S_a (resp. S_b) is the maximal invariant set contained in $\eta_{c,L}^{a}$ (resp. $\eta_{c,L}^{b}$), Lemma 2.3 shows that the index of S_a, denoted by $h(S_a)$, is the same as the index of S_b. This index is easily computed as follows.

LEMMA 2.4. <u>The index of</u> S_b <u>is the pointed 2-sphere (denoted by</u> Σ^2<u>).</u>

<u>Proof.</u> It is easily seen that S_b can be continued to the single hyperbolic rest point $\tilde{P}$ by pulling the curves $M = 0$ and $N = 0$, (case (b)), apart in a manner such that the new vector field equals F near P and $\partial \Sigma_{c}^{b}$ and admits the unique rest point P in Σ_{c}^{b}. It is obvious that $\{\tilde{P}\}$ is the only complete solution of the new equations in $\eta_{c,L}^{b}$. Hence the index of S_b equals the index of $\{\tilde{P}\}$, considered as an isolated invariant set.

The rest point P of F is a node, so that the eigenvalues λ_1 and λ_2 of dF at P are both negative. A simple computation shows that the eigenvalues of $d\Phi$ at $\tilde{P}$ are $\pm\sqrt{-\lambda_i}$, $i = 1,2$. It follows that the index of $\{\tilde{P}\}$ is Σ^2, since the linearized equations are nondegenerate and have two positive eigenvalues. $\square$

We now apply this result to obtain some information about the sets S_a and S_b. Let

$$M_{\epsilon}(X_0) = \{X \in \mathbb{R}^4 : 0 < |X - X_0| < \epsilon\}.$$

THEOREM 2.5. <u>For every</u> $\epsilon > 0$, <u>we have that either</u> $S \cap M_{\epsilon}(\tilde{P}) \neq \emptyset$ <u>or that</u> $S \cap M_{\epsilon}(\tilde{R}) = \emptyset$, <u>where</u> $S = S_a, S_b$.

Proof. If this was not the case, $S_1 = S \setminus \{\widetilde{P}, \widetilde{R}\}$ would be an isolated invariant set; by an easy computation, (see Conley, [3]), we have that

$$(8) \qquad h(S) \cong h(\widetilde{P}) \vee h(\widetilde{R}) \vee h(S_1) \ ,$$

where "$\vee$" is the topological sum of pointed spaces and $\cong$ denotes a homotopy equivalence of such spaces. We obtain a contradiction by computing the second homotopy groups of the left and right hand sides of (8). Since homotopy equivalence induce isomorphisms on the homotopy groups, by Lemma 2.4 and Maunder, [12, Theorem 7.4.6], we have that

$$\mathbb{Z} \cong \pi_2(\Sigma^2) \cong \pi_2(h(\widetilde{P}) \vee h(\widetilde{R}) \vee h(S_1)).$$

However, for two pointed spaces X and Y we have that by Maunder, [12, Theorem 7.2.20],

$$\pi_2(X \vee Y) \cong \pi_2(X) \oplus \pi_2(Y) \oplus \pi_3(X \times Y, X \vee Y),$$

where $\pi_3(Z_1, Z_2)$ is the third relative homotopy group of the pair (Z_1, Z_2). Hence we have that

$$\mathbb{Z} \cong \pi_2(h(\widetilde{P}) \vee h(\widetilde{R})) \oplus \pi_2(h(S_1)) \oplus G \ ,$$

where G is π_3 of the pair

$$(h(\widetilde{P}) \vee h(\widetilde{R}) \times h(S_1), h(\widetilde{P}) \vee h(\widetilde{R}) \vee h(S_1)).$$

However, by the proof of Lemma 2.4 we have that $\Sigma^2 \cong h(\widetilde{P}) \cong h(\widetilde{R})$, and therefore, that

$$\pi_2(h(\widetilde{P}) \vee h(\widetilde{R})) \cong \pi_2(\Sigma^2 \vee \Sigma^2) \cong \mathbb{Z} \oplus \mathbb{Z} \ ,$$

by Maunder, [12, p. 297]. Thus $\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \pi_2(h(S_1)) \oplus G$, yielding a contradiction. $\square$

COROLLARY 2.6. *There exists a nontrivial solution of (4) in* S_α, $\alpha = a, b$, *whose* ω-*limit set consists of one of the rest points* $\widetilde{P}$ *or* $\widetilde{R}$.

Proof. This is an immediate consequence of Theorem 2.5 and the Hartman linearization theorem. For brevity's sake, we omit the details.

3. Hamiltonian systems.

We shall now assume that there exists a function $H(U)$ such that $\nabla H = F$, so that if $E(U,A) = |A|^2 + 2H(U)$, then E is constant on solutions of (4). This will be the case if M and N are chosen as in (3a) or (3b). (More generally, if the zero sets of M and N are $v = k(u)$ and $u = \ell(v)$, respectively, let $\hat{F} = (u(-v^2 + k(u)^2), v(-u^2 + \ell(v)^2))$. Then $\hat{F}$ is a gradient field whose components have the same zero sets as those of F). The solutions, now constrained to lie in a three dimensional manifold, have less room to move around, and substantially more can be proved about their behavior. We shall assume that F is as in case (b), though the following results also hold for certain systems of type (a); this is discussed at the end of this section.

Hamiltonian systems in this context do not occur generically, and their use in mathematical ecology, where the equations are only crudely known, is rather artificial. However, the assumption of such additional structure is reasonable, since the intention here is to prove complexity, rather than simplicity of behavior. More precisely, the sign of the quantity $H(P) - H(R)$ determines which of the two species is the stronger competitor; we will show later that certain qualitative properties of solutions when this quantity is zero persist under various perturbations of the nonlinear term. Furthermore, in a forthcoming paper, we shall obtain travelling wave solutions of quite general classes of equations by constructing a homotopy back to a Hamiltonian system for which $H(P) = H(R)$.

The strategy is to obtain small amplitude periodic solutions (from bifurcation theory) which have values in $\eta^b_{c,L}$. These branches lie in globally defined continua of solutions which also have values in $\eta^b_{c,L}$. This fact, combined with the Hamiltonian structure, can be used to obtain a precise description

of large amplitude periodic solutions, which in turn can be used to obtain heteroclinic or homoclinic orbits depending on $H(P) - H(R)$.

To obtain small amplitude periodic solutions, we consider the equations

(9) $\qquad \lambda^{-2} U_{xx} + F(U) = 0 , \quad U_x(-1) = U_x(1) = 0 ,$

where λ is a large positive parameter. If $U(x)$ is a solution of (9), then $U(\pm \lambda^{-1} x)$ are solutions of (1) on the interval $-\lambda < x < \lambda$, with homogeneous Neumann conditions at the boundary. By translating and "piecing together" such solutions in an obvious manner, we obtain a 2λ-periodic solution of (1), (or of (4)), on $\mathbb{R}^1$. (This procedure yields a genuine solution of (4), as can be seen by applying the uniqueness theorem for o.d.e.'s at points where two solutions are pasted together).

Nontrivial solutions of (9) near Q , (the saddle in Figure 1 (b)), are obtained as follows. The Jacobean of F at Q has one positive and one negative eigenvalue, p and n respectively. The bifurcation points of (9) will occur at values of $\lambda = \lambda_n$ for which $-n^2\pi^2/4\lambda^2 + p = 0$. If e_1 is a unit is a unit vector in $\mathbb{R}^2$ in the direction of the unstable manifold at Q , (for the equations $\dot{U} = F(U)$), we obtain solutions of (9) with $\lambda = \lambda_n + O(\gamma^2)$ of the form

(10) $\qquad U_n(x,\gamma) = Q + \gamma e_1 \sin(n\pi x/2) + O(\gamma^2) ,$

where γ is a small parameter. Let $V_n(x,\gamma)$ be the $2\lambda(\gamma)$-periodic solution of (1) obtained from $U_n(x,\gamma)$ as above.

Now let X be the space of pairs of bounded continuous functions on $(-1,1)$, and let $\& = X \times \mathbb{R}_+$. Then by Rabinowitz's theorem, [18, Theorem 1.3], $(U_n(x,\gamma), \lambda(\gamma))$ lies in a globally defined continuum of solutions $\mathcal{U}_n \subseteq \&$, in the sense that $\mathcal{U}_n$ either "loops back" to the trivial branch $Q \times \mathbb{R}_+$ at some value of $\lambda = \lambda_m$ where $m \neq n$, or $\mathcal{U}_n$ meets ∞ in $\&$. We also obtain from

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U_n in the manner described above a continuum V_n of periodic solutions on $\mathbb{R}^1$ containing $(V_n(x,\gamma),\lambda(\gamma))$. It is easily seen that if $(V,\lambda) \in V_n$, then $(V(x),V'(x)) \in \eta^b_{c,L}$ for all x. From the expansion (10), this clearly holds for (V,λ) near (Q,λ_n). If the assertion fails for some $(V,\lambda) \in V_n$, we can connect (V,λ) to (Q,λ_n) with a continuous curve in V_n. We can then use the continuity of the V-component along this curve, our choice of c and L, and Lemmas 2.1 and 2.2, to obtain a contradiction.

Rabinowitz also characterizes the nodal properties of solutions of nonlinear Sturm-Liouville problems which lie on continua which bifurcate from the trivial solution. However, his argument doesn't have an obvious generalization to systems. We shall prove an analog of this result for systems in the special case under consideration. To do so, we shall need the following lemma.
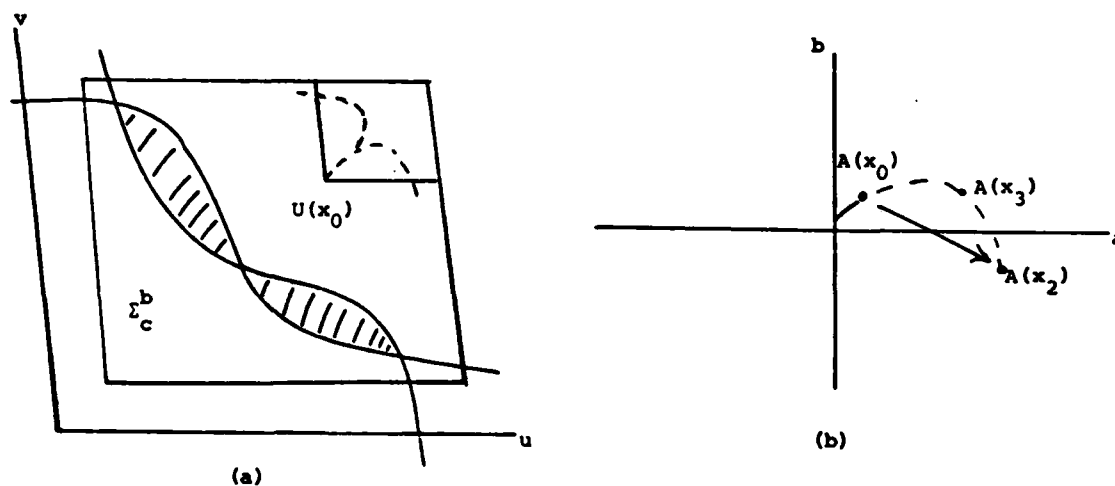

Figure 4

LEMMA 3.1. Suppose that (U,A) is a non-constant solution of (4), that for some $x = x_0$, $U(x_0)$ does not lie in the shaded region in Figure 4(a), and that $A(x_0)$ lies in either the first or the third quadrants of the A-plane, (including $a = 0$ and $b = 0$). Then (U,A) is not a bounded solution of (4) with values in $\eta_{c,L}^b$.

Proof. Suppose that $U(x_0)$ is as above at $x = x_0$. Without loss of generality, we can assume that $U(x_0)$ lies above the shaded region; (it is easily seen that if $U(x_0)$ lies in the extreme upper left - or lower right - hand corners of $\frac{-b}{c}$, the lemma is true regardless of $A(x_0)$). We can also assume that $A(x_0)$ points into the first quadrant; (replace $U(x)$ with $U(-x)$). A moment's reflection shows that we can actually assume that $A(x_0)$ points strictly into the first quadrant.

It clearly suffices to show that $U(x)$ cannot cross the vertical or horizontal lines through $U(x_0)$ for any $x > x_0$. Suppose, for example, that $U(x_1)$ lies on the horizontal line for some $x_1 > x_0$. By the mean value theorem, there must exist x_2 between x_0 and x_1 such that $A(x_2)$ points strictly into the second quadrant of the A-plane. However, the mean value theorem applied to $A(x_0)$ and $A(x_2)$ yields a value x_3 between x_0 and x_2 such that $-F(U(x_3))$ points strictly into the half plane $\{b < 0\}$. However, for U above the shaded region, $-F(U)$ points into the first quadrant, yielding a contradiction. $\square$

LEMMA 3.2. Let L be the straight line in the U-plane passing through Q and through the origin. Then any non-constant element of $\mathcal{U}_1$ crosses L exactly once for $x \in [-1,1]$.

Proof. From the expansion (10), the assertion is clearly true for solutions which are close to (Q,λ_1). Suppose that the lemma is false for some $(U,\lambda) \in \mathcal{U}_1$. We connect (U,λ) to (Q,λ_1) with a continuous curve in $\mathcal{U}_1$. By standard

regularity theorems, we see that U' varies continuously (in the X-topology) along this curve. Hence there must exist an element $(\bar{U}, \bar{\lambda})$ along this branch which intersects L tangentially at least once. If $\bar{V}$ is the $2\bar{\lambda}$-periodic solution corresponding to $\bar{U}$, then a similar statement holds for $\bar{V}$ at some value of $x = \bar{x}$. However, $(\bar{V}(\bar{x}), \bar{V}'(\bar{x}))$ satisfies the hypotheses of Lemma 3.1, so that $(\bar{V}, \bar{V}')$ cannot be a bounded solution of (4) with values in $\eta_{c,L}^b$, yielding a contradiction. $\square$

LEMMA 3.3. <u>The continuum</u> U_1 <u>cannot loop back to the trivial branch at any</u> <u>finite bifurcation point</u> (Q, λ_m), $\lambda_m \neq \lambda_1$, <u>so that</u> U_1 <u>must meet</u> ∞ <u>in</u> $\mathcal{E}$. <u>Moreover, the projection of</u> U_1 <u>on</u> $\mathbb{R}_+$ <u>contains an interval of the form</u> $[\lambda_1, \infty)$.

<u>Proof.</u> The first assertion follows from Lemma 3.2 and the explicit expression (10) for solutions near (Q, λ_m), $m > 1$. The second assertion follows from Rabinowitz's theorem. The last assertion follows from the fact that we have a bound on the projection of U_1 on X, namely, $|U(x)| < \sqrt{2}\,c$ for $(U, \lambda) \in U_1$. $\square$

LEMMA 3.4. <u>Suppose that</u> $(U, \lambda) \in U_1$ <u>and that</u> U <u>is not a constant. Then</u>
(i) $U(\pm 1)$ <u>lies in the interior of the shaded region in Figure 4(a), and</u>
(ii) <u>the components of</u> U <u>are strictly monotone on</u> $(-1,1)$.

<u>Proof.</u> The first assertion follows from the fact that $U_x(-1) = U_x(1) = 0$, and from Lemma 3.1.

We now prove (ii). For definiteness, suppose that $U(1)$ lies in the upper shaded region of Figure 4(a), so that by Lemma 3.2, $U(-1)$ must lie in the lower shaded region. If $U = (u,v)$, let x_0 be the value of x closest to but different from -1 such that $u'(x) = 0$ or $v'(x) = 0$. Suppose that $u'(x_0) = 0$ and that $U(x_0)$ lies on or below L. (If $U(x_0)$ lies above L, then let x_1 be the value of x closest to but different from $+1$ such that

u or v has a critical point; then $U(x_1)$ lies above L , and the following argument can be applied to U at x_1).

Since $U(-1)$ lies in the lower shaded region of Figure 4(a), we have that $u_{xx}(-1) < 0$ and that $v_{xx}(-1) > 0$, so that u has a local maximum and v has a local minimum at $x = -1$. Since u has no critical points between -1 and x_0 , x_0 must either be a point of inflection or a local minimum of u ; that is, $u_{xx}(x_0) = -f(U(x_0)) \geq 0$. However, the region on or below L in which $-f(U) \geq 0$ consists of points that lie above the lower shaded region in Figure 4(a). Moreover, $U'(x_0) = (0, v'(x_0))$ is either vertical or trivial. If V is the periodic solution on $\mathbb{R}^1$ corresponding to U , we have for some $x = \bar{x}$ that $(V(\bar{x}), V'(\bar{x}))$ satisfies the hypotheses of Lemma 3.1. However, (V, V') cannot exit from $\eta_{c,L}^b$; we have therefore obtained a contradiction.   □

THEOREM 3.5. _Suppose that_ $E(\tilde{P}) = E(\tilde{R})$. __Then there exists a heteroclinic orbit__ __of (4) which satisfies (6).__

Proof. Select a sequence $(U_n, \ell_n) \in U_1$ such that $\lim_{n \to \infty} \ell_n = \infty$. Let $x_n \in (-1,1)$ be the unique value of x for which $U_n(x) \in L$. By passing to a subsequence if necessary, we can assume that $\lim x_n = x_0 \in [-1,1]$, that $x_n \leq 0$ for $n = 0,1,2,\ldots$, and that $U_n(1)$ lies above L for all n . If either of the last two statements is reversed, the _proof is similar._

We first show that $U_n(1)$ cannot have a subsequence which approaches Q . Assume that this is not the case and let the subsequence again be denoted by $U_n(1)$. Let $V_n(x)$ be the $2\ell_n$-periodic solution corresponding to U_n . Then

(11) $E(V_n(\ell_n), 0) = E(V_n(-\ell_n), 0)$

for all n . It is easily seen that for any $W \neq Q$ in the shaded region of Figure 4(a), $H(W) > H(Q)$. Thus $V_n(-\ell_n)$ must also approach Q , since if any subsequence approached a point W as above, (11) would be violated for sufficiently large n . Since the components of U_n are monotone on $(-1,1)$, the components of V_n are monotone on $(-\ell_n, \ell_n)$. Thus V_n converges uniformly to Q on $\mathbb{R}^1$.

We obtain a contradiction as follows. Translate $V_n(x)$ to $Z_n(x) = V_n(x + \ell_n x_n)$ so that $Z_n(0) \in L$, and if $\rho_n = \ell_n(1 - x_n) \geq \ell_n$, then $Z_n(x)$ lies above L for all x, $0 \leq x \leq \rho_n$. Let $\Sigma_n = Z_n - Q$, and let $d_n = \text{dist}(Z_n(\rho_n), Q)$. By the monotonicity of the components of Z_n on $[0, \rho_n]$ we have that $Z_n(x)$ lies within a distance of d_n from L, for x in this interval. Now if $M = dF$ at $U = Q$, we have that for $0 \leq x \leq \rho_n$,

$$(12) \qquad \Sigma_n'' + M\Sigma_n = O(\Sigma_n^2) = O(d_n^2).$$

Now let e_1 be the unit vector in $\mathbb{R}^2$ tangent to the unstable manifold of F at Q; we have that e_1 points strictly into the second quadrant and that $e_1^t M = p e_1^t$, where p is the positive eigenvalue of M. Now let $E(x) = e_1^t \sin(\sqrt{p}x)$; multiply (12) by E and integrate from $x = 0$ to $x = \pi/\sqrt{p}$ to obtain

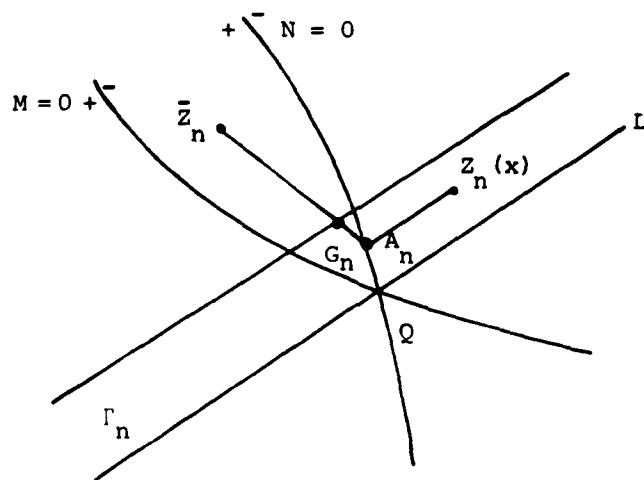$$(13) \qquad -e_1^t \cdot (Z_n(\pi/\sqrt{p}) - Z_n(0)) = O(d_n^2).$$

Figure 5

Note that since $x_n \leq 0$, we have that $\pi/\sqrt{p} < \rho_n$ for large n. Let Γ_n be the set of points lying above L and at a distance of not more than αd_n from L, where $\alpha \in (0, \frac{1}{2})$ is a constant to be chosen as follows. Since the tangents

at Q to $M = 0$ and $N = 0$ both have negative slope, the line connecting a point $\bar{Z}_n$ to G_n can be made to have negative slope by choosing α sufficiently small, (in a manner depending only on the slopes of the above two tangents), where $\bar{Z}_n$ is any point in the upper shaded region in Figure 4(a) at a distance of d_n from L, and G_n is any point in the upper shaded region lying in Γ_n. Thus if θ_n is the angle between $Z_n - G_n$ and e_1, then $\cos\theta_n$ is bounded away from zero independently of n.

Suppose that a subsequence of $Z_n(\pi/\sqrt{p})$ lies outside Γ_n. Let the subsequence again be denoted by $Z_n(\pi/\sqrt{p})$. By the monotonicity assertion of Lemma 3.4 the vector $Z_n(\pi/\sqrt{p}) - Z_n(0)$ points into the second quadrant whereas e_1 points strictly into the second quadrant. Thus if θ_n is the angle between these two vectors, $\cos\theta_n$ is bounded away from zero for all n. Hence the left hand side of (13) is $\mathcal{O}(d_n)$, yielding a contradiction.

We must therefore have that $Z_n(\pi/\sqrt{p})$ lies in Γ_n for all n, and by monotonicity, a similar statement holds for $Z_n(x)$ for all $x \in [0,\pi/\sqrt{p}]$. If $Z_n(x)$ lies inside the upper shaded region, connect $Z_n(x)$ to $\bar{Z}_n = Z_n(\rho_n)$ with a straight line, and let G_n be the point of intersection of this line with the left hand boundary of Γ_n. If $Z_n(x)$ doesn't lie in the upper shaded region, connect $Z_n(x)$ to a point A_n on the closer of the two curves $M = 0$ or $N = 0$ with a line parallel to L, (and hence, in Γ_n). Now connect A_n to $\bar{Z}_n = Z_n(\rho_n)$ with a straight line and let G_n be as above. Since $H_u = f$ and $H_v = g$ are both negative (resp. positive) above (resp. below) the upper shaded region, and since $H_u < 0$ and $H_v > 0$ inside this region, we see from our choice of α, that in both of the above two cases, $H(Z_n(x)) < H(G_n)$ for all $x \in [0,\pi/\sqrt{p}]$. Thus

$$|Z'(x)|^2 = 2H(\bar{Z}_n) - 2H(Z_n(x))$$
$$> 2H(\bar{Z}_n) - H(G_n)$$
$$= 2\nabla H(U_n^*) \cdot (\bar{Z}_n - G_n),$$

where U_n^* is a point on the line connecting G_n to $\bar{Z}_n$. Since Q is a nondegenerate rest point of dF , we have that there exists a constant $K > 0$ such that

$$|\nabla H(U)| = |F(U)| \geq K|U - Q|$$

for U near Q . Moreover, $F(U_n^*)$ points into the second quadrant, and by our choice of α , $\bar{Z}_n - G_n$ points strictly into the second quadrant in a manner such that if θ_n is the angle between $F(U_n^*)$ and $\bar{Z}_n - G_n$ then $\cos\theta_n$ is bounded away from zero for all n . Hence there exists a constant $c > 0$ such that for $0 \leq x \leq \pi/\sqrt{p}$ we have that

$$|Z_n'(x)|^2 \geq c d_n^2 \quad ,$$

since $|U_n^* - Q| \geq \alpha d_n$ and $|Z_n - G_n| \geq (1 - \alpha)d_n$. Moreover, by the monotonicity assertion of Lemma 3.4, we have that $Z_n'(x)$ points into the second quadrant, so that if θ_n is the angle between $Z_n'(x)$ and e_1 , we have that $\cos\theta_n$ is bounded away from zero for all n and $x \in [0, \pi/\sqrt{p}]$. Hence, the left hand side of (13) is once again $O(d_n)$, yielding a contradiction. We must therefore have that $|V_n(\ell_n) - Q| \geq \varepsilon$ for all n and for some $\varepsilon > 0$.

We now obtain the existence of a heteroclinic orbit of (4) connecting $\tilde{P}$ and $\tilde{R}$. By passing to a subsequence if necessary, we can assume that $Z_n(0)$ converges to a limit $\bar{V}$, and that $Z_n'(0)$ converges to a limit $\bar{A}$. Let $(\bar{V}(x), \bar{A}(x))$ be the solution of (4) with data $(\bar{V}, \bar{A})$, and let $J = [0, \bar{x}]$, where $\bar{x} > 0$ is arbitrary. Since $x_n \leq 0$ we have that the components of $Z_n(x)$ must be monotone on J for large n , and since Z_n uniformly approximates $\bar{V}(x)$ on J , the components of $\bar{V}(x)$ must be monotone for all $x > 0$. Hence $\lim_{x \to \infty} \bar{V}(x) = V_\infty$ exists and is finite. Thus the solution $(V_\infty(x), A_\infty(x))$ through any point in the ω-limit set of $(\bar{V}(x), \bar{A}(x))$ must have constant V-component; thus $A_\infty(x) \equiv 0$ and $F(V_\infty) \equiv 0$. Hence $\lim_{x \to \infty} \bar{A}(x) = 0$ and V_∞ is a rest point of F . It follows from the preceding result that $V_\infty \neq Q$. The only other possibility is that $V_\infty = P$.

We finish the proof by showing that $\lim_{x \to -\infty} (\bar{V}(x), \bar{A}(x)) = \tilde{R}$. Let $\sigma_n = (-1 - x_n)$ and suppose that some subsequence of $Z_n(\sigma_n)$ converges to a limit $R_1 \neq R$. From Lemma 3.4, we see that R_1 must lie in the lower shaded region of Figure 4(a). Since $H(R_1) < H(R)$ and $E(Z_n(\sigma_n), 0) = E(Z_n(\rho_n), 0) \to E(\tilde{P}) = E(\tilde{R})$, we obtain a contradiction for sufficiently large n. Thus we have that $\lim_n Z_n(\sigma_n) = R$. Now suppose that $\varliminf_n \sigma_n > -\infty$ and select a subsequence (again denoted by σ_n) such that $\lim_n \sigma_n = \sigma_0 > -\infty$. Since $(Z_n(x), Z_n'(x))$ converges uniformly to $(\bar{V}(x), \bar{A}(x))$ on $[\sigma_0, 0]$, we must have that $\bar{V}(\sigma_0) = R$ and that $\bar{A}(\sigma_0) = 0$, so that $(\bar{V}(x), \bar{A}(x)) \equiv \tilde{R}$, contradiction. Thus $\lim \sigma_n = -\infty$, and the argument of the preceding paragraph can now be applied to obtain

$\lim_{x \to -\infty} (\bar{V}(x), \bar{A}(x)) = \tilde{R}$. $\qquad\qquad\qquad$ □

THEOREM 3.6. Suppose that $E(\tilde{P}) > E(\tilde{R})$. Then there exists a homoclinic orbit of (4) satisfying (5).

Since the details of the proof are similar to those of Theorem 3.5, they will be omitted.

The above arguments can be applied verbatim to systems of type (a) provided that the zero sets of M and N are monotone decreasing with increasing u. This will be the case if the smallest roots of the two parabolas are sufficiently negative, see Figure 6. In a forthcoming paper, we shall obtain travelling wave solutions of systems for which (uM, vN) has such qualitative properties and for which (uM, vN) is not necessarily a gradient.
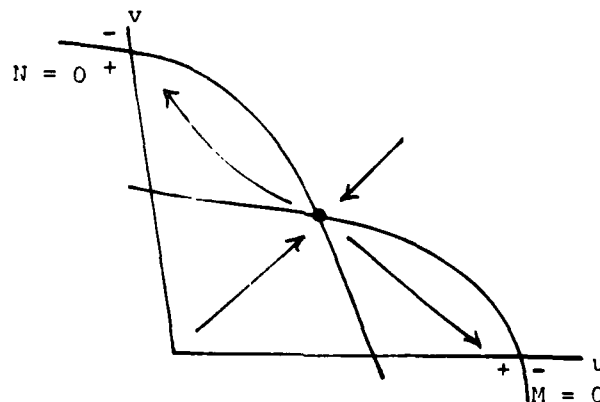

Figure 6

We finally remark that the method of proof of Theorem 3.5 shows that every branch on the continuum V_1 "approaches" a homoclinic or heteroclinic orbit, in the sense described above. Thus the steady state equations (1) admit the familiar "teardrop" of bounded solutions which arises in the case of a scalar equation.

4. Existence of slow travelling waves.

We shall now obtain slow travelling wave solutions of (2) which are in some sense near the stationary heteroclinic orbit connecting $\widetilde{P}$ and $\widetilde{R}$. The technique is to introduce a parameter γ into the equations which is proportional to $E(\widetilde{P}) - E(\widetilde{R})$. In particular, suppose that $H(\widetilde{P}) = H(\widetilde{R})$, where $\nabla H(U) = F(U,0)$, and

$$F(U,\gamma) = (uM(U) + u\gamma , vN(U)) ,$$

where F is as in case (b). We then linearize the travelling wave equations about the above orbit when $\gamma = 0$ and apply the implicit function theorem.

Travelling wave solutions are obtained as follows. Let $W = H^2(\mathbb{R}^1)^2$ and $Y = L^2(\mathbb{R}^1)^2$. Note that the rest points of $F(U,\gamma)$ vary with γ. Let P_γ and R_γ be the rest points "P" and "R" for $F(U,\gamma)$; P_γ and R_γ depend smoothly on γ. Let $\lambda(\xi) = e^\xi(1 + e^\xi)^{-1}$, and define

$$B(\xi,\gamma) = (1 - \lambda(\xi))(R_\gamma - R_0) + \lambda(\xi)(P_\gamma - P_0) .$$

Now let (U,A) be the solution of (4), (6) when $\gamma = 0$, and let $F : \mathbb{R}^2 \times W \times Y$ be defined by

$$F(\gamma,\theta,W) = (W + U + B(\cdot,\gamma))'' + F(W + U + B(\cdot,\gamma),\gamma) - \theta(W + U + B(\cdot,\gamma))' .$$

Note that from the manner in which B was chosen, we have that $F(W + B(\cdot,\gamma)) + (U,\cdot)$ ϵ Y for all $W \epsilon W$. Thus F is a (smooth) mapping of the indicated spaces. Moreover, by construction and the Riemann-Lebesgue lemma, $(\gamma,\theta,\bar{U})$ will be a solution of (6)$_\xi$, (7) if and only if $\bar{U} = W + B(\cdot,\gamma) + U$ where $F(\gamma,\theta,W) = 0$.

Now let $L : \mathbb{R} \times W \to Y$ be the Fréchet derivative of F with respect to (θ, W) evaluated at $(\gamma, \theta, W) = 0$, i.e.,

(14) $L(\theta, W) = W'' + d_U F(U, 0)W - \theta A$,

where $A = U'$. We shall also need to consider the densely defined operator L_0 on Y with domain W defined by

$$L_0 W = W'' + d_U F(U, 0)W .$$

For an operator T, let $\Sigma(T)$ denote the spectrum of T.

LEMMA 4.1. $\Sigma(L_0) \cap \{\sigma \geq 0\}$ <u>consists of isolated eigenvalues of finite</u>

<u>multiplicity</u>.

<u>Proof.</u>   We modify an argument given by Bardos, Matano, and Smoller, [1], for a scalar equation. Let $M(\xi)$ be the matrix valued function $d_U F(U(\xi), 0)$. Then $M(\xi)$ is a symmetric matrix for all ξ, and all the eigenvalues of $M(\pm\infty)$ are strictly negative. Let

$$M_0(\xi) = \lambda(\xi) M(\infty) + (1 - \lambda(\xi)) M(-\infty);$$

it is easily verified that $M_0(\xi)$ is symmetric and has eigenvalues which are strictly less than some negative constant m_0 for all $\xi \in \mathbb{R}^1$. Thus if $R_0 = d^2/d\xi^2 I + M_0(\xi)$, and if for $U, W \in C_0^\infty(\mathbb{R}^1)^2$ we let

$$a_\lambda(W, U) = (R_0 W, U)_Y + \lambda(W, U)_Y ,$$

then a_λ defines a continuous bilinear form on $H^1(\mathbb{R}^1)^2$, and from the above properties of M_0, we have that

$$a_\lambda(W, W) < (\lambda + \max(-1, m_0)) \|W\|_{H^1}^2 .$$

Thus a_λ is coercive on $H^1(\mathbb{R}^1)^2$ for all $\lambda \leq 0$, so that by the Lax-Milgram theorem, $R_0 + \lambda I$ is an isomorphism of $H^1(\mathbb{R}^1)^2$ onto $H^{-1}(\mathbb{R}^1)^2$. Moreover, if $(R_0 + \lambda)U = f \in Y$ then $U \in W$. Thus $R_0 + \lambda$ maps W onto Y so that $-\lambda \notin \Sigma(R_0)$. For such λ we have that $(L_0 + \lambda)(R_0 + \lambda)^{-1} = I + K_\lambda$ where $K_\lambda = (M_0 - M)(R_0 + \lambda)^{-1}$ is an operator on Y. From our choice of M_0 we have that K_λ is compact for $\lambda \leq 0$; clearly $-1 \in \Sigma(K_\lambda)$ if and only if $-\lambda \ \Sigma(L_0)$.

The result follows from the Riess-Schauder theory for compact operators; see [1] for details. □

LEMMA 4.2. <u>The kernel of</u> L_0 <u>is one-dimensional and is spanned by the vector</u> $A(\xi)$. <u>The range of</u> L_0 <u>is the orthogonal complement of</u> A <u>in</u> Y.

<u>Proof.</u> Clearly, $A \in \ker L_0$, so that if λ_0 is the largest element of $\Sigma(L_0)$, we have that $\lambda_0 \geq 0$. Thus by Lemma 4.1, λ_0 is an element of the point spectrum, so that if $Q(W,W) = (L_0 W, W)_Y$, then

$$\lambda_0 = \sup\{Q(W,W) : \|W\|_Y = 1, W \in \mathcal{W}\},$$

and this supremum is attained at a vector $W_0 \in \mathcal{W}$. Suppose that $\lambda_0 > 0$. Then we must have that $(W_0, A)_Y = 0$. However, if W_0 has components σ and τ, then

$$(15) \qquad \lambda_0 = \int_{-\infty}^{\infty} (-\dot{\sigma}^2 - \dot{\tau}^2 + f_u \sigma^2 + 2f_v \sigma\tau + g_v \tau^2) d\xi,$$

where the partials f_u, f_v, and g_v are evaluated at $U(\xi)$. Now $f_v(u,v) \leq 0$ for all $u, v \geq 0$. If σ and τ ever had the same sign and were nonzero for some ξ, we would get a strictly larger value in the integral in (15) by replacing σ with $|\sigma|$ and τ with $-|\tau|$. Thus σ and τ must always have different signs. Moreover, neither σ nor τ ever change sign, since if this was the case at $\xi = \xi_0$, $(|\sigma|, -|\tau|)$ would again be an eigenvector of L_0. However, by the above, $\sigma(\xi_0) = \tau(\xi_0) = 0$; in addition, $\sigma'(\xi_0) = \tau'(\xi_0) = 0$, since $|\sigma|$ and $-|\tau|$ must both be locally H^2 functions. Since $L_0(|\sigma|, -|\tau|) = 0$ is a homogeneous second order system of o.d.e.'s, these conditions imply that σ and τ vanish identically. Thus σ and τ must be of constant and different sign. However, by Lemma 3.4, a similar remark holds for the components of A. Thus $(A, W_0)_Y$ must be nonzero, so that $\lambda_0 = 0$. A similar argument shows that zero is a simple eigenvalue of L_0.

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The last assertion of the lemma follows from the fact that range (L_0) = range $(L_0 k_0^{-1})$ is closed in Y , and the usual relation, $A^\perp = (\ker L_0^*)^\perp = $ cl(range(L_0)) .   □

Now let $W_1 = A^\perp$, and define a mapping $f: \mathbb{R}^3 \times W_1 \to Y$ by

$$f(\gamma, \pi, \theta, W_1) = F(\gamma, \theta, \pi A + W_1) .$$

THEOREM 4.3. <u>There exists a smooth function</u> $h: \mathbb{R}^2 \to \mathbb{R} \times W_1$ <u>where</u> $h(0) = 0$, <u>and</u>

$$h(\gamma, \pi) = (\theta(\gamma, \pi), W_1(\gamma, \pi)) ,$$

<u>such that the set of solutions of</u> $f(\gamma, \pi, \theta, W_1) = 0$ <u>near the origin consists</u> <u>precisely of vectors of the form</u> $(\gamma, \pi, \theta(\gamma, \pi), W_1(\gamma, \pi))$.

Remark. The additional parameter π is present due to the inherent degeneracy in the problem arising from the fact that translations of solutions are solutions. We can therefore "normalize" the space by setting $\pi = 0$, and by the uniqueness assertion of Theorem 4.4 we are assured of obtaining all solutions near the origin modulo translations.

Proof. We have that the derivative of f with respect to (θ, W_1) is precisely the restriction L_1 of L (defined in (14)) to $\mathbb{R} \times W_1$. The theorem will follow from the implicit function theorem if it can be shown that $L_1 : \mathbb{R} \times W_1 \to Y$ is an isomorphism. If $L_1(\theta, W_1) = 0$, we find by multiplying this equation by A and integrating that $-\theta(A,A)_Y = 0$, so that $\theta = 0$, and that $W_1 \in \ker L_0 \cap W_1 = \{0\}$. Hence L_1 is 1:1. Now let $G \in Y$, and put $\theta = -(G,A)_Y/(A,A)_Y$. Then $L_1(\theta, W) = G$ has a solution if and only if

(16) $L_0 W = G + \theta A$

has a solution in W_1 . From our choice of θ and Lemma 4.2, this is obviously the case; thus L_1 is onto .   □

COROLLARY 4.4. $\theta_\gamma(0,0) < 0$.

Proof. Differentiate $F(\gamma, \theta(\gamma, 0), W_1(\gamma, 0)) = 0$ with respect to γ to obtain

$$\Gamma'' + M\Gamma + F_\gamma(U, 0) - \theta_\gamma(0, 0)A = 0 ,$$

where $\Gamma = W_{1\gamma}(0, 0) + B_\gamma(\cdot, 0)$. Note that $F_\gamma(U, 0) = (u, 0)$, where u is the first component of U. Multiply the above equation by A and integrate to obtain

$$\int_{-\infty}^{\infty} A \cdot (u, 0)d\xi - \theta_\gamma(0, 0) \int_{-\infty}^{\infty} |A|^2 d\xi = 0 .$$

Thus, we have that

$$\theta_\gamma(0, 0) = \frac{u^2(\infty) - u^2(-\infty)}{2\|A\|_\gamma^2} < 0 . \qquad \square$$

We remark that all of the above results apply also to the mapping $F_1 = F + \gamma^2 G(W)$ where $G: W \to Y$ is a smooth mapping. Hence travelling wave solutions persist under small, smooth perturbations of the nonlinear term. In particular, we see that such behavior is not peculiar to Hamiltonian systems.

5. Existence of stable clines.

In this section, we introduce a spatially inhomogeneous perturbation into the equations. It is no longer true that translations of solutions are solutions; indeed, for a reasonable class of perturbations, the spectrum of the linearized perturbed equations moves into the stable half plane. Thus even a slight inhomogeneity in the relative competitive strengths of the two species can cause the densities to evolve to a stable large amplitude pattern. This fact has been observed by Peletier, [16], and by Peletier and Fife, [17], for a scalar equation arising in population genetics. Mimura and Nishiura, [14], have obtained small amplitude spatial patterns for a system arising in developmental biology by introducing spatial inhomogenities via perturbed bifurcation theory.

We will assume that $H(P) = H(R)$, where $\nabla H(U) = F(U,0)$ and that F, (case b), has the form

$$(17) \qquad F(U,x,\varepsilon) = (uM(U) + (\varepsilon\rho_1 + \varepsilon^2\phi_1)u, vN(U) + (\varepsilon\rho_2 + \varepsilon^2\phi_2)v) ;$$

here $\rho_i, \phi_i \in L^2(\mathbb{R}^1)$, $i = 1,2$. We shall apply an argument based on the Liapunov-Schmidt procedure to obtain solutions of (1) which also satisfy (5).

We shall need the following theorem.

THEOREM 5.1. <u>Suppose that</u> G <u>is a smooth mapping of a neighborhood of the origin in</u> $\mathbb{R} \times W$ <u>into</u> Y, <u>where</u> W <u>and</u> Y <u>are Banach spaces. Suppose also that</u> $G(0,0) = 0$, <u>and for</u> $(\varepsilon,W) \in \mathbb{R} \times W$ <u>that</u>

(i) $G_W(0,0)H_0 = -G_\varepsilon(0,0)$ <u>for some</u> $H_0 \in W$,

(ii) $\ker G_W(0,0) = \operatorname{span} \{A\}$, <u>for some</u> $A \neq 0$ <u>in</u> W,

(iii)   <u>the range of</u> $G_W(0,0)$ <u>is closed in</u> Y <u>and has codimension one,</u>

(iv)   <u>there exists a one parameter family</u> $W_1(s) \in W$ <u>for small</u> s <u>with</u>
         $W_1(0) = W_{1s}(0) = 0$ <u>such that</u> $G(0, sA + W_1(s)) \equiv 0$,

(v) $G_{\varepsilon W}(0,0)A + G_{WW}(0,0)(H_0, A) \notin \operatorname{range}(G_W(0,0))$.

Then the set of solutions of $G(\varepsilon,W) = 0$ near the origin in $\mathbb{R} \times W$ consists precisely of two transversally intersecting curves. The nontrivial branch has the tangent $(1, H)$, where $H = H_0 + \gamma A$ for some $\gamma \in R^1$ so that it can be parameterized by ε. We therefore obtain a curve $W(\varepsilon) \in W$ such that $W(0) = 0$, $W_\varepsilon(0) = -H$, and $G(\varepsilon, W(\varepsilon)) = 0$.

The proof of a similar theorem can be found in Nirenberg, [15, Theorem 3.2.2]; we shall therefore only sketch the details. Conditions (i) - (iii) and the Liapunov-Schmidt procedure can be used to show that solving $G(\varepsilon,W) = 0$ near the origin in $\mathbb{R} \times W$ is equivalent to solving $G(a,b) = 0$ near the origin in $\mathbb{R}^2$, where $G(a,b) = y^*G(a, aH_0 + bA)$, and y^* is a bounded linear functional on Y which annihilates the range of $G_W(0,0)$. Since $\nabla G(0,0) = (0,0)$, the Morse lemma can be used to obtain the desired result provided that

$$d^2G(0,0) = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_3 \end{pmatrix} \quad \text{is nondegenerate and indefinite, i.e., that } g_1g_3 - g_2^2 < 0.$$

Hypothesis (v) implies that $g_2 \neq 0$ whereas (iv) implies that

$$(18) \qquad g_3 = G_{bb}(0,0) = y^*(G_{WW}(0,0)(A,A))$$

$$= \frac{d^2}{ds^2}(y^*G(0,sA + W_1(s)))\Big|_{s=0}$$

$$= 0 \ .$$

Finally, the tangent of the trivial branch is $(0,A)$; in addition, the tangent of the nontrivial branch at the origin is transverse to that of the trivial branch and lies in the plane spanned by $(1,H_0)$ and $(0,A)$; (see Nirenberg). Hence the nontrivial branch can be parameterized by ε in the manner described above.

Now define a (smooth) mapping $G: \mathbb{R} \times W \to Y$ by

$$G(\varepsilon,W) = (U + W)'' + F(U + W, x, \varepsilon) \ ,$$

where F is as in (17).

THEOREM 5.2. **Suppose that** ρ_1 **and** ρ_2 **are chosen such that**

$$(19) \qquad \int_{-\infty}^{\infty} (a(x)u(x)\rho_1(x) + b(x)v(x)\rho_2(x))dx = 0.$$

Then for an open set of $(\rho_1,\rho_2) \in Y$ **lying in the hyperplane defined by (19) there exists a solution** $W(\varepsilon)$ **of** $G(\varepsilon,W) = 0$, **for sufficiently small** ε .

Proof. We show that the hypotheses of Theorem 5.1 are satisfied. From Lemma 4.2, we see that (19) implies that $G_\varepsilon(0,0) = (u\rho_1,v\rho_2)$ is in the range of $G_W(0,0)$, i.e., $A^\perp$; (ii) and (iii) follow from the same lemma. Condition (iv) is verified by noting that

$$U(x+s) - U(x) = s\, A(x) + s^2[s^{-2}\int_x^{x+s}\int_x^\varepsilon - F(U(\eta))d\eta d\varepsilon];$$

the expression in brackets tends to $-F(U(x))$ uniformly on compact sets as $s \to 0$. Moreover, the derivatives of $U(x)$ are uniformly bounded, and tend to zero exponentially as $|x| \to \infty$. It easily follows that the expression in brackets is uniformly bounded in W as $s \to 0$.

We finally show that (v) holds for most (ρ_1, ρ_2) that satisfy (19).

Note that if $L_0 = G_W(0,0)$, then L_0^{-1} is a bounded self-adjoint linear operator on $A^\perp$ so that $H_0 = L_0^{-1}\begin{pmatrix} u\rho_1 \\ v\rho_2 \end{pmatrix}$ depends linearly and continuously on (ρ_1, ρ_2).

Condition (v) is equivalent to

$$(20) \qquad 0 \neq \int_{-\infty}^{\infty} (a^2 \rho_1 + b^2 \rho_2)dx + \int_{-\infty}^{\infty} A^t d_U^2 F(U(x),0)(A,H_0)dx.$$

Let $V = d_U^2 F(U(x),0)(A,A)$; from (18), we see that $V \in A^\perp$; hence, using the fact that L_0^{-1} is a self-adjoint operator on $A^\perp$, we have that

$$(V,H_0)_Y = \left(L_0^{-1}V, \begin{pmatrix} u\rho_1 \\ v\rho_2 \end{pmatrix} \right)_Y.$$

Moreover, we have that if $\bar{U}(x,s) = U(x+s)$, then

$$\frac{d^2}{ds^2}[\bar{U}_{xx} + F(\bar{U},0)]\Big|_{s=0} = 0,$$

so that if $G = \bar{U}_{ss}(x,0) = -F(U(x))$, we have that

$$L_0 G + d_U^2 F(U,0)(A,A) = 0;$$

that is, $G = -L_0^{-1}V$. Thus $L_0^{-1}V = F(U)$, and (20) is equivalent to

$$(21) \qquad 0 \neq \int_{-\infty}^{\infty} [(a^2 + uf(U))\rho_1 + (b^2 + vg(U))\rho_2]dx.$$

This fails to hold for all (ρ_1, ρ_2) which satisfy (19) if and only if

$$(a^2 + uf, b^2 + vg) = s(ua, vb)$$

for some $s \in \mathbb{R}$. However, in this case at least one of $a^2 - sua$, $b^2 - svb$ must be of constant sign, whereas $-uf(U)$ and $-vg(U)$ both change sign. $\square$

Remark. Since the second order perturbations ϕ_1 and ϕ_2 are free parameters, we see that Theorem 5.2 actually holds for a generic class of L^2 perturbations.

COROLLARY 5.3. Let $\lambda(\varepsilon)$ be the largest eigenvalue of the problem

$$(22) \qquad \Sigma'' + d_U F(U + W(\varepsilon),x,\varepsilon)\Sigma = \lambda\Sigma,$$

and let $\Sigma = \Sigma(\varepsilon)$ be a corresponding eigenvector. Then for all (ρ_1, ρ_2) satisfying (19) and (21), we have that $\lambda_\varepsilon(0) \neq 0$.

<u>Proof</u>. We have that $W_\epsilon(0) = H_0 + \gamma A$ for some $\gamma \in \mathbb{R}$. Let $\Sigma_\epsilon = \Sigma_\epsilon(0)$ and $\lambda_\epsilon = \lambda_\epsilon(0)$. Differentiate (22) with respect to ϵ, set $\epsilon = 0$, multiply by A, and integrate to obtain

$$\lambda_\epsilon = \|A\|_Y^{-2} [\int_{-\infty}^\infty (a^2\rho_1 + b^2\rho_2)\,dx + \int_{-\infty}^\infty A^t d_U^2 F(U,0)(W_\epsilon(0),A)\,dx];$$

we have used the fact that $\lambda(0) = 0$ and that $\Sigma(0) = A$. From (18) and the fact that $W_\epsilon(0) = H_0 + \gamma A$, it can be seen that the expression in brackets equals the right hand side of (21). $\qquad\qquad\square$

Thus for ϵ of the appropriate sign, the spectrum of the linearization of the perturbed equations will lie in the stable half plane. Since the linearized operator is clearly sectorial, standard abstract results can be invoked to obtain the full nonlinear stability of the solutions of the perturbed equations with respect to appropriate perturbations in the initial data of (2); see for example, Henry, [8]. Indeed, such solutions must be asymptotically exponentially stable. We also note that such stable behavior persists under small c^1 perturbations in the nonlinear term; this is an immediate consequence of the implicit function theorem. Even though such perturbations may have to be extremely small, we nevertheless obtain the structural stability of stationary solutions connecting P and R. This fact was not clear from our discussion of autonomous systems in section 3.

6. Concluding Remarks.

Our results indicate that diffusion coupled with a nonlinear competitive interaction can lead to the formation of large amplitude patterns which are locally consistent with the principle of competitive exclusion and yet on a global scale allow both species to persist. (Although such results were proved under the hypothesis of gradient or near-gradient structure, the theorems of section 2, (proved in the absence of such hypotheses), suggest that such behavior is exhibited by solutions of quite robust classes of equations). Furthermore, stability of stationary patterns was achieved through the introduction of spatial

inhomogeneity in the relative competitive strengths of the two species. It is

likely that this is also a necessary condition for stability of such patterns.

In particular, the technique of Bardos and Smoller, [2], can probably be

applied to obtain the instability of the homoclinic orbits and the solutions

of the Neumann problem obtained for autonomous systems in section 3. It would

be interesting to see whether stable solutions of the singularly perturbed

Neumann problem, (9), can be obtained through the introduction of a second,

spatially inhomogeneous perturbation.

Another interesting question is the introduction of time-dependent perturba-

tions. Forced oscillations in competition equations without diffusion were

studied numerically by Koch, [10]. It therefore seems natural to introduce

time-dependent periodic perturbations into systems that admit stable large ampli-

tude stationary solutions. Slowly varying perturbations are likely to give rise

to stable large amplitude periodic solutions, whereas rapidly varying perturba-

tions may give rise to more complicated recurrent behavior.

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Large amplitude solutions are obtained for systems of semilinear reaction-diffusion equations arising in mathematical ecology which describe the evolution of two competing species. Their behavior is locally consistent with the principle of competitive exclusion. Such solutions are first obtained for a special class of steady state equations in which the two species are assumed to be exactly equal competitors; large amplitude patterns for generic classes of equations are then obtained by introducing various perturbations in the relative competitive strengths of the two species. In particular, we obtain (1), travelling wave

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20. Abstract (continued)

solutions through constant perturbations, and (2), stable stationary solutions through

spatially inhomogeneous perturbations.